Aalto University
School of Science
Degree Programme in Engineering Physics and Mathematics

Teemu Päkkilä

Discrete Morse Theory from a Simple-Homotopy Point of View

Aalto University
School of Science

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Technology in the Degree Programme in Engineering Physics and Mathematics

Espoo, October 20, 2014

Supervisor and advisor: Associate Professor Alexander Engström

Aalto University
School of Science
Degree Programme in Engineering Physics and Mathematics

ABSTRACT OF MASTER'S THESIS

Author:	Teemu Päkkilä		
Title: Discrete Morse Theory from a Simple-Homotopy Point of View			
Date:	October 20, 2014	**Pages:**	v + 37
Major:	Mathematics	**Code:**	Mat-1
Supervisor and advisor:	Associate Professor Alexander Engström		

A simple-homotopy equivalence is a refinement of the notion of homotopy equivalence but it is still more general than the notion of homeomorphism. Geometrically simple-homotopy equivalences are generated by collapsing and expanding pairs of cells in adjacent dimensions on the boundary of a CW-complex. These simple-homotopy equivalences are detected by an algebraic invariant called Whitehead torsion. Moreover, the Whitehead torsion is related to another more computable torsion invariant named Reidemeister torsion. We investigate how these torsions are used to study the simple-homotopy type of CW-complexes.

Discrete Morse theory is used to understand how CW-complexes can be simplified without changing the homotopy type. To simplify CW-complexes it uses the same elementary homotopy equivalences that generate simple-homotopy equivalences. However, we must also allow the collapses to be performed within the CW-complex. The question we answer in this thesis is whether allowing these internal collapses results to a more general notion than simple-homotopy equivalence and if so, what happens to the Whitehead torsion. It turns out that these internal collapses can always be turned into a sequence of expansions and collapses resulting to a simple-homotopy equivalence and are therefore also detected by the Whitehead torsion.

Keywords:	Discrete Morse Theory, Simple-Homotopy Theory, Lens Spaces, Reidemeister Torsion, Whitehead Torsion
Language:	English

Aalto-yliopisto
Perustieteiden korkeakoulu
Teknillisen fysiikan ja matematiikan koulutusohjelma

DIPLOMITYÖN TIIVISTELMÄ

Tekijä:	Teemu Päkkilä		
Työn nimi: Yksinkertaiset homotopiaekvivalenssit diskreetissä Morse-teoriassa			
Päiväys:	20. lokakuuta 2014	**Sivumäärä:**	v + 37
Pääaine:	Matematiikka	**Koodi:**	Mat-1
Valvoja ja ohjaaja:	Professori Alexander Engström		

Yksinkertainen homotopiaekvivalenssi on rajoittavampi käsite kuin homotopiaekvivalenssi, mutta kuitenkin yleisempi kuin homeomorfismi. Geometrisesti ajateltuna yksinkertaiset homotopiaekvivalenssit syntyvät poistamalla ja lisäämällä toisiinsa liitettyjä soluja pareittain CW-kompleksin reunalla. Whiteheadin torsio on algebrallinen invariantti, jonka avulla voidaan löytää yksinkertaisia homotopiaekvivalensseja. Lisäksi helpommin määritettävällä Reidemeisterin torsiolla ja Whiteheadin torsiolla on läheinen yhteys, jota käytämme esittääksemme, kuinka näitä torsioita voidaan käyttää CW-kompleksien yksinkertaisen homotopiatyypin määrittämiseen.

Keskeinen tavoite diskreetissä Morse-teoriassa on ymmärtää, kuinka CW-kompleksin solurakennetta voidaan yksinkertaistaa homotopiatyyppiä muuttamatta. Tämän tavoitteen saavuttamiseen diskreetti Morse-teoria käyttää samoja alkeellisia homotopiaekvivalensseja, jotka generoivat yksinkertaiset homotopiaekvivalenssit. Tässä tapauksessa solujen poistaminen täytyy kuitenkin sallia myös CW-kompleksin sisällä. Intuitiivisesti tätä voidaan ajatella romautuksena. Tämän diplomityön päätavoite on tutkia, miten solujen poistaminen CW-kompleksin sisällä vaikuttaa yksinkertaiseen homotopiatyyppiin ja mitä tällöin tapahtuu Whiteheadin torsiolle. Osoittautuu, että CW-kompleksin sisällä tehtävät romautukset voidaan aina korvata yksinkertaisella homotopiaekvivalenssilla, joten Whiteheadin torsio havaitsee myös nämä deformaatiot.

Asiasanat:	Diskreetti Morse-teoria, Yksinkertaisten homotopioiden teoria, Linssiavaruudet, Reidemeisterin torsio, Whiteheadin torsio
Kieli:	Englanti

Acknowledgements

I wish to thank my supervisor and advisor Alexander Engström for suggesting the research topic for this thesis and for the enlightening discussions on the subject. I thank the members of the research group and fellow students for sharing the moments of epiphany and occasional frustration that are so familiar to all mathematicians. Moreover, I want to express my gratitude to my past and present teachers who have guided me to this point all the way from the first pages of the alphabet book.

Finally, I thank my family and friends for the help and support they have shown not only in writing this thesis but also in all other aspects of life.

Thank you.

Espoo, October 20, 2014

Teemu Päkkilä

Contents

Chapter 1

Introduction

One of the central questions in mathematics is how to classify objects. That is, in what sense, say, two topological spaces are the same. The strictest possible condition is to require them to be homeomorphic. Homeomorphic spaces have, in a sense, the same set of points and the same collection of open sets and these are exactly the defining properties of a topological space. The objects that we consider different in everyday life, such as a solid square and a disc, are the same as topological spaces, i.e. there is a homeomorphism between them. The idea behind a homeomorphism is that two homeomorphic spaces can be continuously deformed to each other. However, we know that a disc can be continuously deformed to a point but this is not possible without changing the set of points the disc has – there is no bijection between a disc and a point. There is a more general notion called homotopy equivalence that allows this and it provides a rougher way to divide topological spaces in equivalence classes.

Whitehead's studies in combinatorial topology in the 1940s led to the definition of CW-complexes and to the notion of simple-homotopy equivalence. He tried to represent homotopies between CW-complexes as a composition of simpler moves: elementary collapses and expansions. Using an algebraic K-theory invariant nowadays called the Whitehead torsion he managed to show that not all homotopy equivalences can be decomposed into a composition of these moves. Chapman showed in the 1970s that any homeomorphism has trivial Whitehead torsion and can be presented as a finite sequence of elementary collapses and expansions.

Simple-homotopy theory has a central role in proving important results in surgery theory and polyhedral topology. The Whitehead torsion is used to prove the s-cobordism theorem which plays a central role in classification of high dimensional manifolds. Furthermore, Milnor's counter example to the polyhedral Hauptvermutung can be stated using the Whitehead torsion although he used another torsion element called the Reidemeister torsion. These two torsions have a close connection that we use in Section 3.3 to give an example of a homotopy equivalence that is not simple. For more information on s-cobordisms and the Hauptvermutung, see [Ran96] and [Ran02].

CW-complex structure makes it possible to use algebra to study topological spaces but to be able to make use of CW-structure a topological must first be given one. A common way to do this is to triangulate the space. The problem with triangulation is that it is often too complex and computations become troublesome.

Finding ways to simplify CW-structure was an important motivation when Forman developed a discrete version of Morse theory in the 1990s. Where Morse theory is used to analyse manifolds, discrete Morse theory is used to analyse CW-complexes. The main theorem of discrete Morse theory gives a condition on the minimal number of cells required to build a CW-complex of a certain homotopy type. Given a CW-complex we have to use elmentary collapses to simplify it to the minimal form. However, some of these collapses have to be performed within the complex. The main objective of this thesis is to study how these internal collapses change the simple-homotopy type of the CW-complex. We show in Section 4.2 that they can always be turned into a sequence of elementary expansions and collapses, and the main theorem of discrete Morse theory works in the sense of simple-homotopy.

Chapter 2

Preliminaries

In this chapter we go through some essential definitions required later in the thesis. The chapter is a slightly modified version of a text that has appeared earlier in author's special assignment *Whitehead Torsion and Simple-Homotopy Theory*.

2.1 CW-complexes and Universal Covers

In this section we revise elementary definitions and results in algebraic topology needed in order to understand simple-homotopy theory and discrete Morse theory. The purpose of this section is to set up notation and define only the most important notions. For a more extensive treatment see for example [Hat01] or [Spa89].

First, we make precise what we mean by gluing spaces together. The operation of attaching topological spaces via a continuous map is called *gluing*. Let X and Y be topological spaces and let A be a subspace of Y. Given a continuous map $f : A \to X$, we can attach Y to X to obtain a quotient space

$$X \amalg Y \Big/ f(A) \sim A = X \cup_f Y.$$

Here $X \amalg Y$ denotes the disjoint union of X and Y and $f(A) \sim A$ means that we identify A and its image under f. The topology is given by the quotient topology and the obtained space is called an *adjunction space* and the map f is called an *attaching map*.

We now define the objects to be investigated: the CW-complexes. The building blocks of a CW-complex are called *n-cells*, where n refers to the dimension of the cell. An n-cell is the image of an n-dimensional closed ball under an attaching map. A subspace A of X is called a *discrete space* if every point $x \in A$ has a neighborhood U such that $A \cap U = \{x\}$.

Definition 2.1. A *relative CW-complex* is a pair (X, A) consisting of a Hausdorff space X and a closed subspace A together with a sequence of closed subspaces $\{X^n\}$, called *n-skeletons*, such that

(i) $A \subset X^0$ and $X^0 \backslash A$ is a discrete space. The points of X^0 are *0-cells*;

(ii) for every $n \geq 1$, X^n is obtained by attaching n-dimensional closed balls to X^{n-1};

(iii) $X = \bigcup_n X^n$;

(iv) the topology on X is such that, for every n, a set $S \subset X$ is open if and only if $S \cap X^n$ is open in X^n.

If $A = \emptyset$, then X is a *CW-complex* and if A is a CW-complex, then A is a *subcomplex* of X and we say that (X, A) is a *CW-pair*. If the sequence of closed subspaceses $\{X^i\}$ is finite, then we say that the relative CW-complex (or the CW-complex or the CW-pair, respectively) is *finite*.

When mapping CW-complexes it is useful to preserve the cell structure.

Definition 2.2. Let (X, A) and (X', A') be CW-pairs. A map $f : (X, A) \to (X', A')$ is *cellular* if $f(X^n, A) \subset (X'^n, A')$ for all n.

Attaching a cell is an example of an adjunction space. If e^n is an n-cell it is attached to the $(n-1)$-skeleton X^{n-1} of the CW-complex X. The cells are always attached along their borders and the attaching map of an n-cell e^n is a continuous map $f : \partial D^n \to X^{n-1}$. The map $\varphi : D^n \to X^n$ which coincides with the attaching map f on the border ∂D^n is called a *characteristic map* of the n-cell e^n. The restriction of φ to the interior of D^n is a homeomorphism. If all the characteristic maps are homeomorphisms also on the border, then the CW-complex is *regular*.

Proposition 2.3. *Let X and Y be CW-complexes and A a subcomplex of X. Let f be a map $A \to Y$ and q the quotient map $X \amalg Y \to X \cup_f Y$. If f is cellular, then $X \cup_f Y$ has a natural CW-structure with the cells of Y attached via their original attaching maps together with the cells of $X \backslash A$ attached via $q \circ \varphi_i$, where the φ_i are the original attaching maps.*

Many useful spaces arise by gluing spaces together. We denote the unit interval $[0, 1]$ by I. If $f : X \to Y$ is a map between topological spaces, the mapping cylinder M_f is the space $(X \times I) \cup_f Y$, where f is seen as a map $X \times \{0\} \to Y$.

A *covering space* of X is a space $\widetilde{X}$ together with a continuous map $p : \widetilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ with the property that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets, each homeomorphic to U_α via p. Automorphisms $\widetilde{X} \to \widetilde{X}$, called *covering transformations*, form a group $G(\widetilde{X})$ under composition. If a covering space of X is simply connected, i.e. path-connected with a trivial fundamental group, it is called the *universal cover* of X. A connected CW-complex always admits a (unique) universal cover.

The *group ring* $R[G]$ of a group G over a ring R is the set of linear combinations

$$\sum_{g\in G} r_g g$$

where $r_g \in R$ for every $g \in G$, and where only finitely many of the coefficients r_g are non-zero. Summation and multiplication by a scalar $r \in R$ in $R[G]$ are defined by extension. Multiplication is defined

$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} r_h h\right) = \sum_{g,h\in G} (r_g r_h)(gh).$$

Observe that under these operations $R[G]$ is a ring.

The group of covering transformations $G(\widetilde{X})$ of a universal cover is isomorphic to $\pi_1(X, x_0)$. Thus the fundamental group $\pi_1(X, x_0)$ acts freely on the universal cover of X by covering transformations and this action induces an action of $\mathbb{Z}[\pi_1(X, x_0)]$ on $\widetilde{X}$. The cellular chain complex of universal covers consists of abelian groups

$$C_n(\widetilde{X}, \widetilde{Y}) \cong H_n(\widetilde{X}^n, \widetilde{X}^{n-1}) \cong F_{\mathbf{Ab}}(\{e^n \mid e^n \text{ is an } n\text{-cell of } \widetilde{X}\}),$$

where $F_{\mathbf{Ab}} : \mathbf{Grp} \to \mathbf{Ab}$ is the free abelian group functor. We can see the abelian groups $C_n(\widetilde{X}, \widetilde{Y})$ as $\mathbb{Z}[\pi_1(X, x_0)]$-modules under the action of $\mathbb{Z}[\pi_1(X, x_0)]$ on $\widetilde{X}$.

2.2 K-Theory

We begin by setting up some notation and defining $K_1(R)$. We denote the group of $n \times n$-matrices over a ring R by $M_n(R)$ and the general linear group of degree n over a ring R by $GL_n(R)$. The direct limit of the sequence

$$GL_1(R) \hookrightarrow GL_2(R) \hookrightarrow \cdots$$

is called the *infinite dimensional linear group* over a ring R and it is denoted $GL(R)$. The group operation is given by ordinary matrix multiplication.

A matrix is called an *elementary transvection* if it coincides with the identity matrix save for one off-diagonal element. An elementary transvection is denoted $\tau_{i,j}(r)$, where $r \in R$ is the off-diagonal element on the row i and column j. Observe that $\tau_{i,j}(-r) = \tau_{i,j}^{-1}(r)$ and elementary transvections are invertible. The identity is given by the identity matrix. The group generated by $n \times n$-elementary transvections over R is denoted by $E_n(R)$ and it forms a subgroup of $GL_n(R)$. Again, we get a sequence of inclusions

$$E_1(R) \hookrightarrow E_2(R) \hookrightarrow \cdots$$

and the direct limit of the sequence is the *infinite dimensional group of elementary transvections* $E(R)$. A lemma by Whitehead states that $E(R)$ is the commutator subgroup of $GL(R)$. ([Mag02, (9.7)])

Definition 2.4. The *first algebraic K-theory group* $K_1(R)$ of a ring R is the abelianization $GL(R)/E(R)$ of $GL(R)$.

Besides as a product of matrices, the group operation in $K_1(R)$ can also be seen as a *block sum.*

Proposition 2.5. *Let $A \in GL_n(R)$ and $B \in GL_m(R)$ and let X be an $(n \times m)$-matrix and Y an $(m \times n)$-matrix. Then*

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = [A][B]$$

in $K_1(R)$.

For a proof, see [DK01, Theorem 11.8]. In the usual linear algebra we require the entries of a matrix to belong to some field $\Bbbk$. Replacing a field by a ring arouses questions like whether the determinant still exists. Let $A, B \in M_n(R)$. The following are the characteristic properties of determinant:

- $\det(AB) = \det(A)\det(B)$.
- $\det(\tau_{i,j}(x)) = 1$ for every $0 \leq i, j \leq n$, $i \neq j$.
- for every $n \geq 1$
$$\det\left(\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}\right) = \det(A).$$
- A is invertible if and only if $\det(A)$ is a unit.

If the ring R is commutative, the properties of determinant are still true so it makes sense to consider the special linear group $SL(R)$ consisting of the matrices in $GL(R)$ with determinant 1.

Definition 2.6. The *special Bass-Whitehead group* $SK_1(R)$ of a commutative ring R is the quotient $SL(R)/E(R)$.

Proposition 2.7. *Let R be a commutative ring. Then*

$$K_1(R) \cong SK_1(R) \oplus R^*.$$

For a proof, see [Mag02, (9.18)]. By Proposition 2.7 the computation of $K_1(R)$ for commutative rings reduces to computing $SK_1(R)$ and the group of units.

To give an idea of how the computations look, we compute $K_1(R)$ for a euclidean ring. Recall that a commutative ring R is called *euclidean* if there are no zero-divisors and there exists a function $f : R \to \mathbb{N}$ such that for every $r, r' \in R$

- $f(r) = 0$ if and only if $r = 0$.
- $f(rr') = f(r)f(r')$.
- if $r' \neq 0$, there exist $p, q \in R$ such that $r = pr' + q$ and $0 \leq f(q) < f(r')$.

Proposition 2.8. *For a euclidean ring R, $K_1(R) \cong R^*$.*

Proof. First observe that multiplication by an elementary transvection corresponds to row-addition. Denote $A = (a_{i,j}) \in GL_n(R)$, where R is a euclidean ring. As A is invertible, we have $a_{k,1} \neq 0$ for some k. Let $f : R \to \mathbb{N}$ be a function satisfying the conditions of the definition of a euclidean ring. We can write $1 = pa_{k,1} + q$, where $0 \leq f(q) < f(a_{k,1})$. If $f(a_{k,1}) = 1$, we have $f(q) = 0$, so $q = 0$ and $a_{k,1}$ is a unit. If $f(a_{k,1}) > 1$, then $a_{k,1}$ is not a unit, and thereby generates a proper ideal $\langle a_{k,1} \rangle \subset R$. However, A is invertible so $\langle a_{i,1} \mid 1 \leq i \leq n \rangle = R$. Hence there exists $l \neq k$ such that $a_{l,1} \notin \langle a_{k,1} \rangle$ and we can write $a_{l,1} = pa_{k,1} + q$, where $f(q) > 0$. Now substracting the k^{th} row multiplied by p from the l^{th} row changes the entry $a_{l,1}$ to q, and by the definition of a euclidean ring we know that $f(q) < f(a_{k,1})$. We now have a way to decrease the value of $f(a_{i,1})$, where $a_{i,1}$ is a non-zero element in the first column. Iterating the procedure allows us to reduce the proof to the case where there are only units and 0's in the first column.

From now on, assume that every non-zero entry in the first column of A is a unit and let $a_{k,1}$ be the first such entry. Then multiplicating A by $\tau_{1,k}(1)\tau_{k,1}(-1)\tau_{1,k}(1)$ moves the entry to the first row of the column 1. Next, multiplications by $\tau_{i,1}\left(-a_{i,1}a_{1,1}^{-1}\right)$, $i > 1$, transform all the other entries in the first column to zero. We have transformed A to the form

$$\begin{bmatrix} a_{11} & * \\ 0 & A' \end{bmatrix},$$

where a_{11} is a unit and $A' \in GL_{n-1}(R)$. Iterating the whole process for A' and so on allows us to transform A to an upper triangular matrix whose diagonal entries are either units or 0's.

Now that the matrix is in upper triangular form we can use the same idea as for columns to transform all the entries on the first row, save for the first one, to 0's. A similar procedure can be performed on every row to obtain a diagonal matrix $D = (d_{i,j})$. Note that since all the transformations have been done by elementary transvections, the obtained diagonal matrix D has the same determinant as the original matrix. Every non-zero entry of D is a unit on the diagonal so $D \in GL_n(R)$.

Let $B = (b_{i,j})$ be a $n \times n$-matrix over R. By Proposition 2.5, the matrices

$$\begin{bmatrix} I_n & B \\ 0 & I_n \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} I_n & 0 \\ B & I_n \end{bmatrix}$$

are generated by elementary transvections and therefore belong to $E_{2n}(R)$. Now consider an invertible matrix $C \in GL_n(R)$. We see that a matrix of the form

$$\begin{bmatrix} C & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} I_n & C \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C^{-1} & I_n \end{bmatrix} \begin{bmatrix} I_n & C \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

is in $E_{2n}(R)$.

Next, consider again the obtained diagonal matrix D. Using transformations of the form

$$\begin{bmatrix} I_{n-m-2} & 0 & 0 & 0 \\ 0 & a_{ii} & 0 & 0 \\ 0 & 0 & a_{ii}^{-1} & \\ 0 & 0 & 0 & I_m \end{bmatrix} \in E_n(R),$$

where $m \leq n-2$, we can transform D to a form, where the only non-identity entry on the diagonal is $a_{1,1}$, which is a unit. To summarize, we began with a matrix $A \in GL_n(R)$, where R is a euclidean ring. We ended up having a diagonal matrix D', whose only non-identity entry is a unit in the $(1,1)$-position. All this was obtained by using only elementary transvections. Hence $\det(D') = \det(A)$. So, if $A \in SL_n(R)$, then $\det(D') = 1$ and $D' = I_n$. We deduce that $A \in E_n(R)$. Since this holds for every n, we have $SL(R) \subset E(R)$. On the other hand it is clear that $E(R) \subset SL(R)$ so $SL(R) = E(R)$. Therefore $SK_1(R) = \{0\}$. We conclude by Proposition 2.7.

□

This result is a hint why we should think of $K_1(R)$ as a determinant. If $\mathbb{k}$ is a field, then $K_1(\mathbb{k}) \cong \mathbb{k}^*$. Indeed, in $K_1(\mathbb{k})$ the matrices in $GL(\mathbb{k})$ are divided in equivalence classes according to their determinant.

Example 2.9. $\mathbb{Z}$ is a euclidean ring with respect to the absolute value and $K_1(\mathbb{Z}) \cong \mathbb{Z}^* = \{\pm 1\}$ by Proposition 2.8.

Chapter 3

Simple-Homotopy Theory

Simple-homotopy equivalence is a refinement of the notion of homotopy equivalence given by composing certain elementary homotopy equivalences. An algebraic invariant called the Whitehead torsion is used to detect these simple-homotopy equivalences. In this section we go through the main definitions and results of simple-homotopy theory. The classic textbook reference in this subject is [Coh73] and this chapter is mainly a survey on certain parts of the book.

In the first section we define simple-homotopy equivalence geometrically and prove some useful results related to simplifying a CW-complex without changing its simple-homotopy type. Often it is more useful to consider the Whitehead torsion instead of constructing the simple-homotopy equivalence explicitly. We will not consider the questions behind the definition of the Whitehead torsion and we take the relation of the Whitehead torsion and the geometric definition of simple-homotopy equivalence as given. Instead, we will see how to use the Whitehead torsion to investigate the relation of homotopy equivalence and simple-homotopy equivalence.

3.1 Collapses and Expansions

Homotopy equivalence is an equivalence relation defined for general topological spaces. Simple-homotopy equivalence is a notion of homotopy equivalence that takes into account the CW-decomposition of a space. The original motivation behind simple-homotopy theory was to see if homotopy equivalent spaces could be detected by composing certain simple moves: elementary collapses and elementary expansions.

Definition 3.1. Let (X, A) be a finite CW-pair such that $X = A \cup e^n \cup e^{n+1}$ and the cells e^n and e^{n+1} are not in A. Then we say that the inclusion $A \hookrightarrow X$

is given by an *elementary collapse* or X collapses to A by an elementary collapse if the following holds. Write $\partial D^{n+1} = D^n_+ \cup D^n_-$ with $\partial D^n_+ = \partial D^n_-$. The $(n+1)$-cell e^{n+1} has a characteristic map $\varphi : D^{n+1} \to X$ such that $\varphi\big|_{D^n_+}$ is a characteristic map for the n-cell e^n and $\varphi(D^n_-) \subset A$. If the inclusion $A \hookrightarrow X$ is given by an elementary collapse, we denote $X \searrow_e A$ or $A \nearrow^e X$, respectively. The latter one is then called an *elementary expansion*.

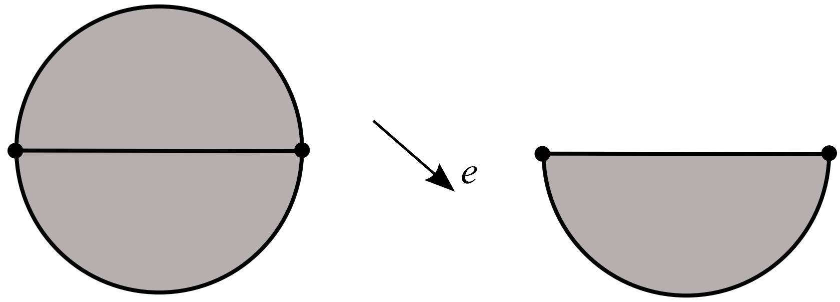

Figure 3.1 A CW-complex with two 0-cells, three 1-cells and two 2-cells collapsing to a CW-complex with two 0-cells, two 1-cells and one 2-cell by an elementary collapse.

Theorem 3.2. *Let (X, A) be a CW-pair such that $X = A \cup e^n \cup e^{n+1} \searrow_e A$. Then there is a cellular deformation retraction $d : X \to A$ such that $d\big|_{X\times\{1\}}(e^n \cup e^{n+1}) \subset \varphi(D^n_-)$, where φ is the characteristic map of e^{n+1} and D^n_- is the lower hemisphere of ∂D^{n+1} that is mapped into A by φ.*

Proof. Since the inclusion $A \hookrightarrow X$ is given by an elementary collapse we have, by definition, that the characteristic map $\varphi : D^{n+1} \to X$ of the $(n+1)$-cell e^{n+1} maps the lower hemisphere D^n_- into A. Denote $\varphi\big|_{D^n_-} = \psi$. Now it is easy to see that $A \cup_\psi D^{n+1}$ and X are homeomorphic. Furthermore, identifying D^{n+1} with $D^n \times I$ so that $D^n \times \{0\}$ corresponds to D^n_- and the rest of the border of the cylinder corresponds to D^n_+ shows that $A \cup_\psi D^{n+1}$ is homeomorphic to the mapping cylinder of ψ. Since the mapping cylinder deformation retracts to A by sliding each point $(x, i) \in D^n \times I$ to $\psi(x)$ we see that all the points are sent into $\varphi(D^n_-) \subset A^n$. On A the deformation retraction is identity so it is cellular. □

By Theorem 3.2 we know that to any elementary collapse we can associate a deformation retraction d. On the other hand, for every elementary expansion, there is an inclusion and this inclusion is the homotopy inverse of d. It follows that if two spaces have the same simple-homotopy type, then there is a map $f_1 \circ \cdots \circ f_n$ between them and each f_i is either a deformation retraction or an inclusion. This composition is also a homotopy equivalence since both deformation retraction and its homotopy inverse are homotopy equivalences.

Definition 3.3. Let (X, A) be a CW-pair. We say that X *collapses to* A or A *expands to* X and denote $X \searrow A$ or $A \nearrow X$ if there are subcomplexes $X_1, X_2, \ldots, X_n$ such that

$$X \searrow_e X_1 \searrow_e \cdots \searrow_e X_n \searrow_e A.$$

Definition 3.4. A finite sequence of operations, each of which is either an elementary collapse or an elementary expansion, is called a *formal deformation*.

Definition 3.5. Let X and Y be CW-complexes with a formal deformation between them. Denote $f : X \to Y$, $f = f_1 \circ f_2 \circ \cdots \circ f_n$, where each f_i is either the deformation retraction of an elementary collapse or the inclusion of an elementary expansion. Any map $X \to Y$ which is homotopic to f is said to be a *simple-homotopy equivalence*. If there is a simple-homotopy equivalence $X \to Y$, the spaces X and Y are said to be *simple-homotopy equivalent* and we denote $X \nearrow\!\!\searrow Y$.

Definition 3.6. Let (X, A) and (Y, A) be CW-pairs. If X and Y are simple-homotopy equivalent and the composition $f_1 \circ f_2 \circ \cdots \circ f_n$ realizing the formal deformation restricts to the identity on A, then we say that X and Y are simple-homotopy equivalent relative to A.

Since the map that realizes the formal deformation is a homotopy equivalence, a simple-homotopy equivalence is also a homotopy equivalence and defines an equivalence relation. It is reasonable to ask whether the reverse holds – that is, if a homotopy equivalence is a simple-homotopy equivalence. It turns out this is not the case. In fact, if two spaces are homotopy equivalent, they are not simple-homotopy equivalent in general. Thus simple-homotopy equivalence is a refinement of the notion of homotopy equivalence.

The following result relating the simple-homotopy type of a CW-complex and the attaching maps used to construct it will be useful later.

Theorem 3.7. *Let $X_0 = A \cup e_0$ and $X_1 = A \cup e_1$ be CW-complexes, where the e_i are n-cells with characteristic maps $\varphi_i : D^n \to X_i$ such that $\varphi_0\big|_{\partial D^n}$ and $\varphi_1\big|_{\partial D^n}$ are homotopic. Then X_0 and X_1 are simple-homotopy equivalent relative to A.*

Proof. The idea of the proof is illustrated in Figure 3.2. Assume that $e_0 \cap e_1 = \emptyset$. Then $X = A \cup e_0 \cup e_1$ is a well-defined CW-complex with X_0 and X_1 as subcomplexes. Now, denote the homotopy from $\varphi_0\big|_{\partial D^n}$ to $\varphi_1\big|_{\partial D^n}$ by f. In other words, let $f : \partial D^n \times I \to A$ with $f(x,i) = \varphi_i\big|_{\partial D^n}$ for $i = 0,1$. By the cellular approximation theorem the map

$$f : (\partial D^n \times I, \partial D^n \times \{0,1\}) \to (A, A^{n-1})$$

is homotopic to a map g such that $g\big|_{\partial D^n \times \{0,1\}} = f\big|_{\partial D^n \times \{0,1\}}$ and $g(\partial D^n \times I) \subset A^n$. Define $\varphi : \partial(D^n \times I) \to X$ by setting $\varphi\big|_{\partial D^n \times I} = g$ and $\varphi\big|_{D^n \times \{i\}} = \varphi_i$ for $i = 0,1$. Identify $D^n \times I$ with D^{n+1}. We get a new CW complex $Y = X \cup e^{n+1}$ by attaching a new $(n+1)$-cell e^{n+1} to X via φ.

We claim that

$$X_0 \nearrow^e Y \searrow_e X_1.$$

Denote $\partial D^{n+1} = D^n_+ \cup D^n_-$. For the expansion, identify $D^n \times \{1\}$ with D^n_+ and the rest of $\partial(D^n \times I)$ with D^n_-. Then $\varphi\big|_{D^n_+}$ is the characteristic map for e_1 and $\varphi(D^n_-) \subset X_0$, so Y is obtained from X_0 by an elementary expansion. Doing the identifications the other way round we get the elementary collapse from Y to X_1. Moreover, A remains intact during the whole process so this finishes the proof for the case $e_0 \cap e_1 = \emptyset$.

If $e_0 \cap e_1 \neq \emptyset$, we can construct a CW-complex $X_0' = A \cup e_0'$ such that $e_0' \cap (e_0 \cup e_1) = \emptyset$ and such that e_0' has the same attaching maps as e_0. The CW-complexes X_0 and X_0' are then simple-homotopy equivalent by the proof of the case $e_0 \cap e_1 = \emptyset$. $\square$

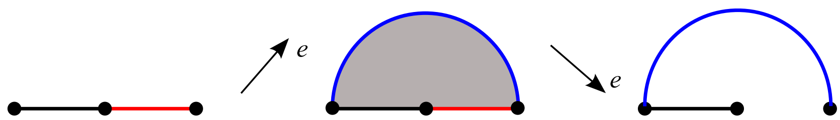

Figure 3.2 The picture illustrates the idea behind the proof. In this case the cell e_1 is the red cell and the cell e_2 the blue one. The black 1-cell gives the homotopy of the attaching maps.

3.2 Whitehead Torsion

In this section we take a little detour and investigate simple-homotopy from an algebraic point of view. It will not be necessary for the rest of the thesis but it is useful when studying simple-homotopy theory.

The Whitehead torsion is an algebraic way to detect simple-homotopy equivalence. It is an element in a quotient of the first algebraic K-theory group of a certain ring associated to a topological space.

Definition 3.8. Let G be a multiplicative group. The quotient

$$K_1(\mathbb{Z}[G]) \big/ \langle g^{\pm 1} \mid g \in G \rangle$$

is the *Whitehead group* of G denoted $Wh(G)$.

A group homomorphism $f : G \to G'$ induces a ring homomorphism

$$f_* : \mathbb{Z}[G] \to \mathbb{Z}[G']$$

given by

$$f_* \left(\sum_i n_i g_i \right) = \sum_i n_i f(g_i),$$

which in turn induces a map

$$Wh(f_*) : Wh(G) \to Wh(G')$$

given by

$$Wh(f_*)([(a_{i,j}]) = [(f_*(a_{i,j}))].$$

It is easy to check that $Wh(f \circ g) = Wh(f) \circ Wh(g)$ and $Wh(\mathrm{Id}_G) = \mathrm{Id}_{Wh(G)}$. This means that we have a covariant functor $Wh : \mathbf{Gr} \to \mathbf{Ab}$.

Let $C_\bullet$ be an acyclic chain complex of finitely generated free modules over a ring R with a canonical choice of basis. It can be shown that an acyclic chain complex is contractible with a chain contraction $s_\bullet$. Using this chain contraction we obtain an isomorphism of R-modules

$$(\partial_\bullet + s_\bullet)\big|_{C_{2i+1}} : \bigoplus_i C_{2i+1} \to \bigoplus_i C_{2i}.$$

Definition 3.9. The image of $(\partial_\bullet + s_\bullet)\big|_{C_{2i+1}}$ in

$$K_1(R) \big/ \langle \pm 1 \rangle$$

is the *torsion* of the chain complex $C_\bullet$ and it is denoted $\tau(C_\bullet)$.

We define the Whitehead torsion for a finite connected CW-pair (X, A) such that A is a deformation retract of X. For a generalization to the non-connected case, see [Coh73, §19]. Consider the universal covers $\widetilde{X}$ and $\widetilde{A}$. It can be shown that $\widetilde{A}$ is a deformation retract of $\widetilde{X}$. The chain complex $C_\bullet(\widetilde{X}, \widetilde{A})$ is an acyclic chain complex of $\mathbb{Z}[\pi_1(X)]$-modules and the modules have a basis labelled by the cells of (X, A).

Definition 3.10. The *Whitehead torsion* of the pair (X, A) is the image of

$$\tau(X, A) = \tau(C_\bullet(\widetilde{X}, \widetilde{A}))$$

in $Wh(\pi_1(A))$.

Consider a general homotopy equivalence $f : X \to Y$. If f is not cellular, replace it by a cellular homotopy equivalence given by the cellular approximation theorem. The mapping cylinder M_f is a finite CW-complex because f is cellular and it deformation retracts to X since f was assumed to be a homotopy equivalence. The homotopy equivalence f induces an isomorphism of fundamental groups so, by functoriality, we have a map $f_* : Wh(\pi_1(X)) \to Wh(\pi_1(Y))$.

Definition 3.11. The *Whitehead torsion* of a homotopy equivalence $f : X \to Y$ is $\tau(f) = f_*\tau(M_f, X)$.

Theorem 3.12. *Let $f : X \to Y$ and $g : Y \to Z$ be cellular homotopy equivalences. Then*

(i) *If $f \simeq g$, then $\tau(f) = \tau(g)$;*

(ii) *$\tau(g \circ f) = \tau(g) + g_*\tau(f)$.*

For a proof, see [Coh73, §22]. By Theorem 3.12 the torsion $\tau(f)$ of a homotopy equivalence does not depend on the choice of cellular approximation, because if $\tilde{f}$ and $\tilde{f}'$ are different cellular approximations, then $\tilde{f} \simeq \tilde{f}'$ and $\tau(\tilde{f}) = \tau(\tilde{f}')$. The use of the Whitehead torsion lies in the following theorem by Whitehead.

Theorem 3.13. *A homotopy equivalence f of CW-complexes is a simple-homotopy equivalence if and only if $\tau(f)$ is trivial.*

The proof of it is rather involved and not presented here. It can be found in [Coh73, (22.2)]. We illustrate its use with a couple of examples. First of all, assuming that there are non-trivial Whitehead groups, the Whitehead torsion shows that not all homotopy equivalences are simple.

Lemma 3.14 ([Coh73, (22.3)])**.** *Let (X, A) be a CW-pair and A a deformation retract of X. Then $\tau(\iota) = \iota_*\tau(X, A)$, where $\iota : A \to X$ is the inclusion.*

The following construction is from [Tur01, Proposition 7.1].

Theorem 3.15. *Let Y be a CW-complex and let $\tau_0 \in Wh(\pi_1(Y))$. Then there is a CW complex X and a homotopy-equivalence $f : X \to Y$ such that $\tau(f) = \tau_0$.*

Proof. Assume X is a CW-complex such that Y is a deformation retract of X and $\tau(X, Y) = -\tau_0$. Let f be the homotopy inverse of the inclusion $\iota : Y \to X$. Then

$$\tau(f) = -f_*\tau(\iota)$$

because $0 = \tau(\mathrm{Id}_Y) = \tau(f \circ \iota) = \tau(f) + f_*\tau(\iota)$, where we used Theorem 3.12 and Theorem 3.13. Moreover,

$$-f_*\tau(\iota) = (-f_* \circ \iota_*)\tau(X, Y) = -\tau(X, Y) = \tau_0$$

by Lemma 3.14.

It remains to show that there always exists a CW-complex X such that Y is a deformation retract of X and $\tau(X, Y) = -\tau_0$. Since $-\tau_0 \in Wh(\pi_1(Y))$, it is represented by an invertible matrix $(a_{i,j}) \in GL_k(\mathbb{Z}[\pi_1(Y)])$. Fix an integer $n \geq 2$. Define Y' by attaching k copies of n-spheres to Y at a point $y \in Y$. Since $n \geq 2$, we must have $\pi_1(Y', y) \cong \pi_1(Y, y)$, so $\pi_n(Y', y)$ is a $\mathbb{Z}[\pi_1(Y)]$-module. Consider the elements $[S^n_j] \in \pi_n(Y', y)$ represented by the k attached n-spheres. Define

$$[\varphi_i] = \sum_{j=1}^{k} a_{i,j}[S^n_j]$$

for every $i \in \{1, \ldots, k\}$. Each $[\varphi_i] \in \pi_1(Y', y)$ is represented by a map $\varphi_i : S^n \to Y'$. Define a CW-complex

$$X = Y' \cup_{\varphi_1} D_1^{n+1} \cup_{\varphi_2} \cdots \cup_{\varphi_k} D_k^{n+1}.$$

The chain complex $C_\bullet(\widetilde{X}, \widetilde{Y})$ has only two non-trivial modules, namely

$$C_n(\widetilde{X}, \widetilde{Y}) \cong C_{n+1}(\widetilde{X}, \widetilde{Y}) \cong \mathbb{Z}[\pi_1(Y)]^k.$$

If we choose the orientations and lifts of the attached n- and $(n + 1)$-cells appropriately, the boundary morphism ∂_{n+1} is represented by $(a_{i,j})$. But then $[(a_{i,j})] = \tau(X, Y) = -\tau_0$. Since $(a_{i,j})$ is invertible, the chain complex $C_\bullet(\widetilde{X}, \widetilde{Y})$ is acyclic and Y is a deformation retract of X. □

A simple-homotopy equivalence must have a trivial Whitehead torsion by Theorem 3.13 and, on the other hand, we just saw that any element in the Whitehead group of a fundamental group of a CW-complex can be realized as a torsion of a homotopy equivalence. There are fundamental groups with a non-trivial Whitehead group, for example $\pi_1(L(7, q)) = \mathbb{Z}_7$ and $Wh(\mathbb{Z}_7) \neq \{0\}$, so there must be homotopy equivalences with a non-trivial Whitehead torsion. We note that in many cases the Whitehead group is trivial and, indeed, for the most ordinary spaces there is no difference between a homotopy equivalence and a simple-homotopy equivalence. For example, if G is trivial; free abelian; or cyclic of order 2, 3, 4 or 6; then $Wh(G) = \{0\}$. In general, it is not an easy task to compute Whitehead groups. There is a well known conjecture that the Whitehead group of any torsion-free group should vanish but this has been open for decades. See [KL05, Conjecture 21.16].

The question whether there is a simple-homotopy equivalence $X \to Y$ when there is a homotopy equivalence is a bit more difficult. It depends not only on the Whitehead group of the fundamental group of the space in question but also on the group of equivalence classes of self-homotopy equivalences of X under homotopy.

Proposition 3.16. *Let X be a CW-complex. Any CW-complex Y homotopy equivalent to X is simple-homotopy equivalent to X if and only if*

$$Wh(\pi_1(X)) = \{\tau(f) \mid f \textit{ is a self-homotopy equivalence on } X\}.$$

Proof. Fix Y. If the CW-complexes X and Y are homotopy equivalent, define

$$S_Y = \{\tau(f) \mid f : Y \to X \text{ is a homotopy equivalence}\}.$$

By definition $S_Y \neq \emptyset$ so consider $\tau(f) \in S_Y$. Assume there is also a CW-complex Z that is simple-homotopy equivalent to Y. Then there is a simple-homotopy equivalence $s : Z \to Y$ and $\tau(f \circ s) \in S_Z$. By Theorems 3.12 and 3.13

$$\tau(f \circ s) = \tau(f) + f_*\tau(s) = \tau(f),$$

so $S_Z \subset S_Y$. By symmetry, $S_Y \subset S_Z$. We have shown that if Y and Z have the same simple-homotopy type, then $S_Y = S_Z$.

The converse holds as well. If $S_Y = S_Z$, then we can choose homotopy-equivalences $f : Y \to X$ and $g : Z \to X$ such that $\tau(f) = \tau(g)$. Let $h : X \to Z$ be the homotopy inverse of g and consider the composition $h \circ f : Y \to Z$. By Theorem 3.12,

$$\tau(h \circ f) = \tau(h) + h_*\tau(f) = -h_*\tau(g) + h_*\tau(f) = 0.$$

In conclusion, homotopy equivalent spaces X and Y are simple-homotopy equivalent if and only if $S_X = S_Y$.

Observe that S_X is exactly the set of the Whitehead torsions of self-homotopy equivalences on X. Moreover, by definition $S_X \subset Wh(\pi_1(X))$. Assume that any CW-complex Y that is homotopy equivalent to X is also simple-homotopy equivalent to X. Any element $\tau_0 \in Wh(\pi_1(X))$ can be realized as $\tau(f)$ for some homotopy equivalence $f : Y \to X$ by Theorem 3.15. Then Y and X are also simple-homotopy equivalent by assumption and by what we just proved $S_Y = S_X$. But $\tau_0 = \tau(f) \in S_Y = S_X$ so $Wh(\pi_1(X)) \subset S_X$. For the converse, assume $Wh(\pi_1(X)) = S_X$. For any CW-complex Y we have $S_Y \subset Wh(\pi_1(X))$ and therefore $S_Y \subset S_X$ by assumption. Since $S_X = Wh(\pi_1(X))$, any $\tau_0 \in S_X$ can be realized as $\tau(f)$ for some CW-complex Y and some homotopy equivalence $f : Y \to X$ and so $\tau_0 = \tau(f) \in S_Y$. Therefore we have $S_Y = S_X$ for any Y homotopy equivalent to X and by what we proved, X and Y are simple-homotopy equivalent. $\square$

Lastly, we note how simple-homotopy equivalences relate to homeomorphisms. The following highly non-trivial result is due to Chapman.

Theorem 3.17 ([Cha74, Theorem 1])**.** *Any homeomorphism of compact connected CW-complexes is a simple-homotopy equivalence.*

3.3 Lens Spaces

We now use Theorem 3.13 to give an example of a homotopy equivalence that is not simple.

Definition 3.18. Consider the unit sphere $S^3 \subset \mathbb{C}^2$. Let p and q be coprime integers and define a free $\mathbb{Z}_p$-action on S^3 by

$$(z_1, z_2) \mapsto (\zeta z_1, \zeta^q z_2),$$

where ζ is the p^{th} root of unity. A *lens space* $L(p, q)$ is the quotient of S^3 by this action.

Recall from the previous section that $Wh(\mathbb{Z}_7)$ is non-trivial. So, if we have a space whose fundamental group is $\mathbb{Z}_7$ we should be able to find a homotopy equivalence that is not simple. But $\pi_1(L(7, q)) = \mathbb{Z}_7$ so the lens spaces $L(7, q)$ are the spaces that we start investigating.

The following theorems give the homotopy classification of lens spaces, credited to Franz [Fra35, Fra43], Rueff [Rue38] and Whitehead [Whi41].

Theorem 3.19. *There is a map of lens spaces $f : L(p, q) \to L(\tilde{p}, \tilde{q})$ such that $f_*(g) = \tilde{g}^a$ if and only if*

$$q \deg(f) \equiv a^2 \tilde{q} \pmod{p},$$

where f_* is the induced map of the fundamental groups.

Theorem 3.20. *A map of lens spaces $f : L(p, q) \to L(\tilde{p}, \tilde{q})$ is a homotopy equivalence if and only if* $\deg f = \pm 1$.

For a proof, see for example [Coh73, (29.5)] or [DK01, Theorem 11.35].

Now consider the spaces $L(7, 1)$ and $L(7, 2)$. By choosing $a = 2$ the condition for the existence of a degree 1 map $f : L(7, 1) \to L(7, 2)$ is satisfied. By Theorem 3.20 this map is a homotopy equivalence. To see if the map is a simple-homotopy equivalence, we have to compute its Whitehead torsion.

The problem with the Whitehead torsion is that it is extremely difficult to compute. However, there is another similar element, called Reidemeister torsion, that is more computable. It is defined in a similar fashion as the Whitehead torsion and since we have already defined the Whitehead torsion, we can use it to define the Reidemeister torsion.

Definition 3.21. Let $C_\bullet$ be a finite chain complex of finitely generated S-modules with a canonical choice of basis and let $f : S \to R$ be a ring homomorphism to a commutative ring. Suppose that $C_\bullet \otimes_S R$ is acyclic. Then

$$\tau_f(C_\bullet) = \det(\tau(C_\bullet \otimes_S R)) \in R^*/\langle \pm 1 \rangle$$

is called the *Reidemeister torsion* of $C_\bullet$ with respect to f.

The advantage of the Reidemeister torsion is that in many cases it can be defined even though the Whitehead torsion cannot be. This is because $C_\bullet \otimes_S R$ can be acyclic even though $C_\bullet$ is not. We use multiplicative notation for the Reidemeister torsion.

Definition 3.22. Let X be a CW-complex and R be a commutative ring. Suppose $f : \mathbb{Z}[\pi_1(X)] \to R$ is a ring homomorphism such that the chain complex

$$C_\bullet(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R$$

is acyclic. Then

$$\tau_f(X) = \tau_f(C_\bullet(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R) \in R^*/\langle \pm f(\pi_1(X)) \rangle$$

is the *Reidemeister torsion* of X with respect to f.

The image of $\pm \pi_1(X)$ is quotiented out to make $\tau_f(X)$ independent of the orientation of the cells and choice of lifts.

Theorem 3.23 ([Tur01, Theorem 9.1])**.** *Let $f : X \to Y$ be a homotopy equivalence of finite connected CW-complexes. Suppose that $\rho : \mathbb{Z}[\pi_1(Y)] \to R$ is a ring homomorphism to a commutative ring (with* *unity) so that* $C_\bullet(\widetilde{Y}) \otimes_{\mathbb{Z}[\pi_1(Y)]} R$ *is acyclic. Set* $\eta = \rho \circ f_* : \mathbb{Z}[\pi_1(X)] \to R$*. Then* $C_\bullet(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R$ *is acyclic and*

$$\rho_*(\tau(f)) = \tau_\rho(Y)\tau_\eta(X)^{-1}.$$

The proof of the following proposition draws from [DK01, Section 11.6] and [Nic03, Example 2.9].

Proposition 3.24. *There is a ring homomorphism*

$$f : \mathbb{Z}[\pi_1(L(p,q))] \to \mathbb{C}$$

such that the twisted chain complex $C_\bullet(\widetilde{L(p,q)}) \otimes_{\mathbb{Z}[\pi_1(L(p,q))]} \mathbb{C}$ *is acyclic. Moreover, the Reidemeister torsion of* $L(p,q)$ *with respect to* f *is*

$$\tau_f(L(p,q)) = (\zeta - 1)(\zeta^r - 1),$$

where $\zeta \neq 1$ *is the* p^{th} *root of unity and* $qr \equiv 1 \pmod{p}$*.*

Proof. Let g be the generator of $\mathbb{Z}_p$. A lens space $L(p,q)$ has a well known CW-structure with one cell in each dimension. For every $k \in \mathbb{Z}_p$, we set

$$\begin{aligned}
e^0_k &= (\zeta^k, 0)\\
e^1_k &= \left\{(e^{i\theta}, 0) \mid \frac{k2\pi}{p} \leq \theta \leq \frac{(k+1)2\pi}{p}\right\}\\
e^2_k &= \left\{(z_1, s\zeta^k) \in \mathbb{C}^2 \mid s \in \mathbb{R}, \sqrt{|z_1|^2 + s^2} = 1\right\}\\
e^3_k &= \left\{(z_1, z_2) \in S^3 \mid \frac{k2\pi}{p} \leq \arg(z_2) \leq \frac{(k+1)2\pi}{p}\right\}.
\end{aligned}$$

For $0 \leq n \leq 3$, the cells e^n_k define a $\mathbb{Z}_p$-equivariant cell structure on S^3. That is, each n-cell is mapped to an n-cell by the free $\mathbb{Z}_p$-action

$$g \cdot (z_1, z_2) = (\zeta z_1, \zeta^q z_2).$$

Indeed, it is a straightforward computation to check that

$$\begin{aligned}
g \cdot e^0_k &= e^0_{k+1},\\
g \cdot e^1_k &= e^1_{k+1},\\
g \cdot e^2_k &= e^2_{k+q},\\
g \cdot e^3_k &= e^3_{k+q}.
\end{aligned}$$

The orientation on the cells is defined to satisfy $\partial(e_k^2) = \sum_{j=0}^{p-1} e_j^1$, $\partial(e_k^1) = e_{k+1}^0 - e_k^0$ and $\partial(e_k^3) = e_{k+1}^2 - e_k^2$. Let $qr \equiv 1 \pmod p$. It follows that

$$\begin{aligned}
\partial(e_k^1) &= e_{k+1}^0 - e_k^0 = g \cdot e_k^0 - e_k^0 = (g-1)e_k^0, \\
\partial(e_k^2) &= \sum_{j=0}^{p-1} e_j^1 = \left(\sum_{j=0}^{p-1} g_j \right) e_0^1, \\
\partial(e_k^3) &= e_{k+1}^2 - e_k^2 = g^r \cdot e_k^2 - e_k^2 = (g^r - 1)e_k^2.
\end{aligned}$$

To summarize, we have a $\mathbb{Z}_p$-equivariant cell structure on S^3 consisting of p cells in each dimension. Since $L(p,q)$ is the quotient of S^3 by the $\mathbb{Z}_p$-action in question, we obtain a cell structure for $L(p,q)$ with one cell in each dimension. The cellular chain complex of the universal cover S^3 is

$$0 \to \mathbb{Z}[\mathbb{Z}_p] \xrightarrow{g^r-1} \mathbb{Z}[\mathbb{Z}_p] \xrightarrow{\sum_{j=0}^{p-1} g_j} \mathbb{Z}[\mathbb{Z}_p] \xrightarrow{g-1} \mathbb{Z}[\mathbb{Z}_p] \to 0. \tag{†}$$

Observe that this chain complex has the homology of S^3 so it is not acyclic. To compute the Reidemeister torsion, consider the ring homomorphism $f : \mathbb{Z}[\mathbb{Z}_p] \to \mathbb{C}$ given by $f(g) = \zeta$. Tensor the sequence (†) with $\mathbb{C}$ over $\mathbb{Z}[\mathbb{Z}_p]$ and denote the obtained chain complex by $C_\bullet$. Let e be a cell of $L(p,q)$. The chain complex $C_\bullet$ is a chain complex of free $\mathbb{Z}[\mathbb{Z}_p]$-modules, where each module C_n has a basis of the form $e \otimes 1$. But $L(p,q)$ has only one cell in each dimension, so C_n is isomorphic to $\mathbb{C}$ for every $0 \leq n \leq 3$ and trivial otherwise.

Now $f(g^r - 1) = \zeta^r - 1$, $f(g-1) = \zeta - 1$ and

$$f\left(\sum_{j=0}^{p-1} g^j \right) = \sum_{j=0}^{p-1} f(g^j) = \sum_{j=0}^{p-1} \zeta^j \equiv 0 \pmod p.$$

The chain complex $C_\bullet$ becomes

$$0 \to \mathbb{C} \xrightarrow{\zeta^r-1} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{\zeta-1} \mathbb{C} \to 0$$

and is acyclic. Now $\tau(C_\bullet)$ is represented by a matrix of the form

$$\begin{bmatrix} \zeta - 1 & 0 \\ s_1 & \zeta^r - 1 \end{bmatrix},$$

where $s_1 : C_1 \to C_2$ is a map in the chain contraction. The Reidemeister torsion is

$$\tau_f(C_\bullet) = \det(\tau(C_\bullet)) = (\zeta - 1)(\zeta^r - 1).$$

□

Earlier in this section we noted that there is a homotopy equivalence

$$f : L(7,1) \to L(7,2)$$

of degree one. We also saw that this map fulfils the conditions of Theorem 3.19 for $a = 2$, so $f_*(g) = \tilde{g}^2$ for $g \in \pi_1(L(7,1))$ and $\tilde{g} \in \pi_1(L(7,2))$. Let

$$\eta : \pi_1(L(7,1)) \to \mathbb{C}^*,$$

and

$$\rho : \pi_1(L(7,2)) \to \mathbb{C}^*$$

be group homomorphisms such that $\eta(g) = \zeta$ and $\rho(\tilde{g}) = \tilde{\zeta}$, where ζ is a 7^{th} root of unity and extend them to ring homomorphisms to $\mathbb{C}$. In order to apply Theorem 3.23, we require $\eta(g) = \rho(f_*(g))$, which leads to the condition $\zeta = \tilde{\zeta}^2$ so we choose $\tilde{\zeta} = \zeta^4$. By Theorem 3.23

$$\rho_*(\tau(f)) = \tau_\rho(L(7,2))\tau_\eta(L(7,1))^{-1}.$$

By Proposition 3.24

$$\tau_\eta(L(7,1)) = (\zeta - 1)^2$$

and

$$\tau_\rho(L(7,2)) = (\tilde{\zeta} - 1)(\tilde{\zeta}^4 - 1) = (\zeta^4 - 1)(\zeta^2 - 1).$$

But then

$$\rho_*(\tau(f)) = \frac{(\zeta^4 - 1)(\zeta^2 - 1)}{(\zeta - 1)^2} \neq 1$$

since

$$|(\zeta - 1)^2| \neq |(\zeta^4 - 1)(\zeta^2 - 1)|.$$

This proves that $\tau(f) \neq 0$ and f is not a simple-homotopy equivalence by Theorem 3.13. Moreover, f cannot be a homeomorphism by Theorem 3.17.

Chapter 4

Discrete Morse Theory

Discrete Morse theory was developed in [For98] and it is, as the name suggests, a discrete approach to the ideas behind original Morse theory which is used to analyze the topology of a manifold. As such, discrete Morse theory involves a notion of Morse function. The discrete Morse functions are lists assigning a number to each cell of a CW-complex and they give a method to simplify the complex. In this chapter we investigate how these simplifications relate to the simple-homotopy theory presented earlier in the thesis.

4.1 Central Definitions and Results

A central result behind discrete Morse theory is that given a CW-complex X and an n-cell e^n with an attaching map $\varphi : \partial D^n \to X$, the homotopy type of $X \cup_\varphi D^n$ only depends on the homotopy type of X and the homotopy class of φ.

Theorem 4.1 ([LW69, Chapter IV, Corollary 2.4])**.** *Let $h : X \to X'$ be a homotopy equivalence of topological spaces with $\varphi_1 : \partial D^n \to X$ and $\varphi_2 : \partial D^n \to X'$ attaching maps of n-cells. If $h \circ \varphi_1 \simeq \varphi_2$ then $X \cup_{\varphi_1} D^n$ and $X' \cup_{\varphi_2} D^n$ are homotopy equivalent.*

The theorem has an important corollary when h is the identity map.

Corollary 4.2. *Let X be a topological space with $\varphi_1 : \partial D^n \to X$ and $\varphi_2 : \partial D^n \to X'$ attaching maps of n-cells. If $\varphi_1 \simeq \varphi_2$ then $X \cup_{\varphi_1} D^n$ and $X \cup_{\varphi_2} D^n$ are homotopy equivalent.*

Topological manifolds of dimensions two and three can always be given a CW-structure by triangulation. It is difficult, but possible, to come up with non-triangulable examples in higher dimensions. However, it is often troublesome to work with triangulations and the more general CW-complexes tend to have fewer cells. For example, a minimal triangulation of a torus in Figure 4.1 consists of seven vertices, 21 edges and 14 triangles whereas the CW-decomposition in Figure 4.2 has only one 0-cell, two 1-cells and one 2-cell. Given a finite CW-complex it would be useful to have a way to find an equivalent CW-decomposition with fewer cells. Making use of Theorem 4.1, discrete Morse theory provides such a way up to a homotopy.

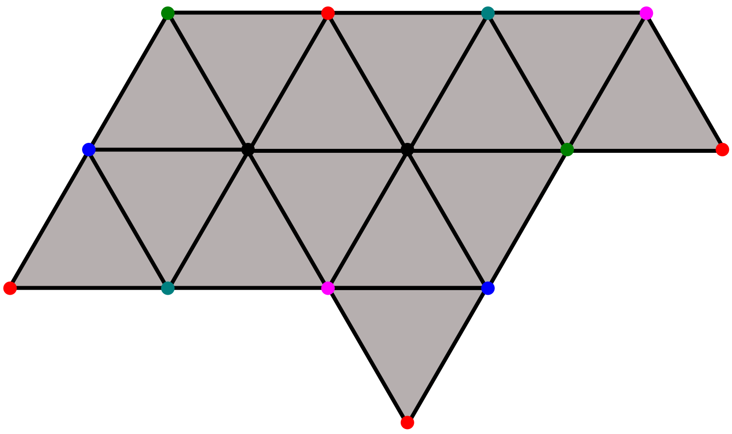

Figure 4.1 A minimal triangulation of a torus, where the boundaries are identified according to the colours of the vertices.

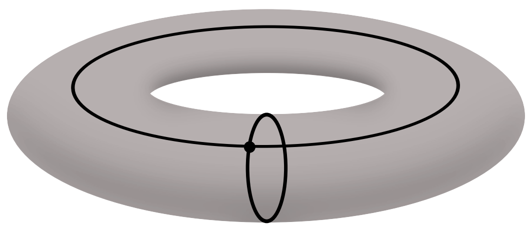

Figure 4.2 A minimal CW-decomposition of a torus.

Before going further we set up some terminology and notation. Let e^n and e^{n+1} be cells in a CW-complex X. We write $e^n < e^{n+1}$ or $e^{n+1} > e^n$ when $e^n \subset \overline{e^{n+1}}$, where $\overline{e^{n+1}}$ is the closure of e^{n+1}. Observe that e^n has to lie on the boundary of $\overline{e^{n+1}}$. We say that e^n is a *face of* e^{n+1}. Sometimes we might not state the dimension of the cells explicitly. In that case, it makes sense to write $e \leq d$ if $e < d$ or $e = d$ and $d \geq e$, respectively. If the characteristic map $\varphi : D^{n+1} \to X$ restricts to a homeomorphism $\varphi\big|_{\varphi^{-1}(e^n)}$ and $\overline{\varphi^{-1}(e^n)}$ is a closed n-ball, then e^n is a *regular face of* e^{n+1}. Otherwise, e^n is an *irregular face*.

We proceed to introduce some aspects of discrete Morse theory as done in [For98] and [For02].

Definition 4.3. Let X be a finite CW-complex and let K denote the set of open cells of X with K^n the cells of dimension n. A *discrete Morse function* on X is a function $f : K \to \mathbb{R}$ such that for every cell in X we have the following:

(i) If e^n is an irregular face of e^{n+1}, then $f(e^{n+1}) > f(e^n)$. Moreover,

$$\#\left\{e^{n+1} > e^n \mid f(e^{n+1}) \leq f(e^n)\right\} \leq 1.$$

(ii) If e^{n-1} is an irregular face of e^n, then $f(e^{n-1}) < f(e^n)$. Moreover,

$$\#\left\{e^{n-1} < e^n \mid f(e^{n-1}) \geq f(e^n)\right\} \leq 1.$$

Definition 4.4. Let f be a discrete Morse function on X. An n-cell $e^n \in K^n$ is a *critical cell of index* n if

$$\#\left\{e^{n+1} > e^n \mid f(e^{n+1}) \leq f(e^n)\right\} = 0$$

and

$$\#\left\{e^{n-1} < e^n \mid f(e^{n-1}) \geq f(e^n)\right\} = 0.$$

An n-cell can only be a critical cell of index n.

Example 4.5. Consider a disc with a CW-structure as in Figure 4.3. If we

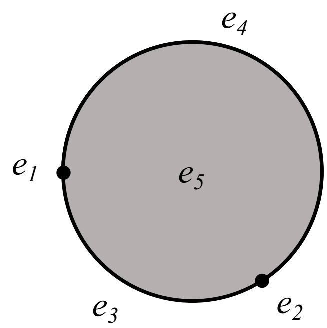

Figure 4.3

define a function f by

$$f(e_1) = 0,\ f(e_2) = 1,\ f(e_3) = 2,\ f(e_4) = 1,\ f(e_5) = 3,$$

then f is a discrete Morse function with three critical cells: e_1 is a critical cell of index 0, e_3 of index 1 and e_5 of index 2. We could as well define

$$g(e_1) = 0,\ g(e_2) = 1,\ g(e_3) = 1,\ g(e_4) = 2,\ g(e_5) = 2,$$

and g would also be a discrete Morse function but with only one critical cell: e_1!

Definition 4.6. Let f be a discrete Morse function on a CW-complex X. For $c \in \mathbb{R}$ define the *level subcomplex* of X as

$$X(c) = \bigcup_{\substack{e \in K \\ f(e) \leq c}} \bigcup_{d \leq e} d.$$

Using level subcomplexes one can prove the discrete Morse theory equivalents of the main theorems of smooth Morse theory. Here we consider only regular CW-complexes but the results generalize for general (finite) CW-complexes as well. This will be discussed in the following section.

Lemma 4.7 ([For98, Theorem 3.3])**.** *Let f be a discrete Morse function on a regular CW-complex X. If $a < b$ are real numbers such that the cells e with $f(e) \in (a, b]$ are not critical, then $X(b)$ collapses to $X(a)$.*

Lemma 4.8 ([For98, Theorem 3.4])**.** *Let f be a discrete Morse function on a regular CW-complex X and suppose e^n is the only critical cell with $f(e^n) \in (a, b]$. Then there is a continuous map $\varphi : \partial D^n \to X(a)$ such that $X(b)$ is homotopy equivalent to $X(a) \cup_\varphi D^n$.*

For our purposes it is enough to consider the idea behind the proofs. Rigorous proofs are presented in [For98]. Consider the CW-complex in Example 4.5 with a discrete Morse function f, but without the 2-cell e_5. Denote this CW-complex by X. Now $X(0) = e_1$ and $X(1) = e_1 \cup e_2 \cup e_4$. There are no critical cells with values of the discrete Morse function in $(0, 1]$ and $X(1)$ collapses to $X(0)$ just as Theorem 4.7 predicts. The next step is the complex $X(2) = X(1) \cup e_3$. From the picture we can see that this is not an expansion, but instead we attach a cell to $X(1)$. Considering the discrete Morse function, the difference is that in the interval $(1, 2]$ we now have a value assigned to a critical cell. This is motivates Lemma 4.8. The homotopy type does not change because an elementary collapse is a deformation retraction by Theorem 3.2. Theorem 4.1 now states that we can perform the collapses given by Lemma 4.7 within the complex without altering the homotopy type. In the following proposition we do not make the regularity assumption otherwise imposed in this discussion.

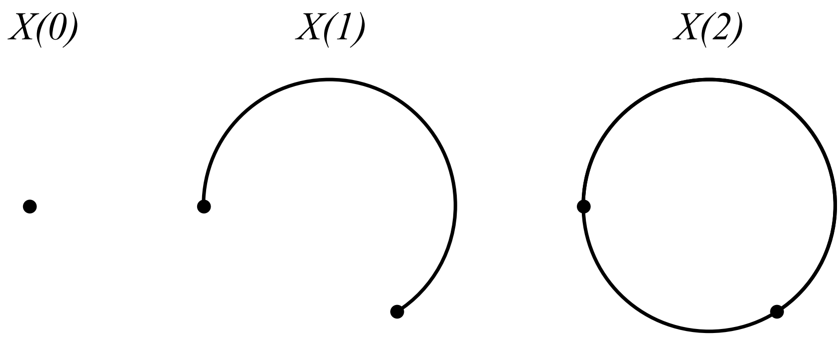

Figure 4.4

Proposition 4.9. *Let (X, A) be a CW-pair such that $X = A \cup e^m \cup e^{m+1} \searrow_e A$. If $\varphi : \partial D^n \to X$ is an attaching map of an $n-$cell, then there is a continuous map $\psi : \partial D^n \to A$ such that φ and ψ are homotopic and if $\varphi(x) \in e^m \cup e^{m+1}$ for some $x \in \partial D^n$, then $\psi(x) \in \operatorname{Im}(\eta)$, where η is the attaching map of e^{m+1}. In particular, there is a homotopy-equivalence $X \cup_\varphi D^n \to A \cup_\psi D^n$.*

Proof. For an idea of the proof, see Figure 4.5. If $\varphi(\partial D^n) \subset A$, then there is nothing to prove since an elementary collapse is a deformation retraction by Theorem 3.2.

By assumption $X = A \cup e^m \cup e^{m+1}$ where the characteristic maps of the cells e^m and e^{m+1} fulfil the conditions of Definition 3.1. Consider the case where D^n is not attached to A, that is $\varphi(\partial D^n) \subset X$ but $\varphi(\partial D^n) \not\subset A$. Observe that in this case we must have $n > m$. Consider the deformation retraction $d : X \times I \to X$ of the elementary collapse. By definition, d is continuous. Denote the composition $d\big|_{X\times\{1\}} \circ \varphi$ by ψ. By Theorem 3.2 we have $d\big|_{X\times\{1\}}(e^m \cup e^{m+1}) \subset \operatorname{Im}(\eta) \cap A \subset A^m$. But we must have $n > m$ so $\psi : \partial D^n \to A$ is a well-defined attaching map for an n-cell. The composition $d \circ \varphi$ gives a homotopy from φ to ψ, so $X \cup_\varphi D^n$ and $A \cup_\psi D^n$ are homotopy equivalent by Theorem 4.1. □

In conclusion, what a discrete Morse function essentially does, is that it pairs cells in adjacent dimensions. These cells are then not critical and can be collapsed. The cells that are not paired are critical and cannot be collapsed.

Theorem 4.10 ([For98, Theorem 10.2])**.** *Let X be a CW-complex with a discrete Morse function f and denote the number of critical cells of f of index n by m_n. Then X is homotopy equivalent to a CW-complex with m_n cells of dimension n for each n.*

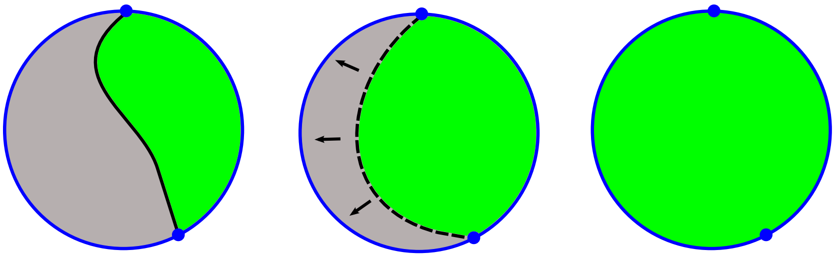

Figure 4.5 The homotopy equivalence of Proposition 4.9. The green cell is the attached n-cell and the blue subcomplex is the CW-complex A.

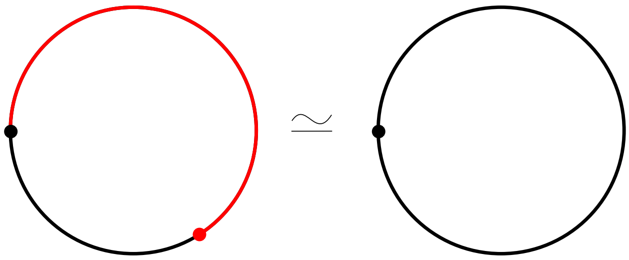

Figure 4.6

If the sublevel complexes are CW-complexes (which is the case for regular CW-complexes), then Theorem 4.10 follows directly from Lemmas 4.7 and 4.8. Starting with the CW-complex $X = X(2)$ in what was discussed above, there is no way to get rid of both of the 1-cells without altering the homotopy type. Nonetheless, removing one 1-cell and one 0-cell would not make a difference. This removal is exactly the elementary collapse Lemma 4.7 predicts and the cells that are left are the critical ones.

Example 4.11. In Example 4.5 we had a discrete Morse function with one critical cell in each dimension. By Theorem 4.10 the CW-complex in question is homotopy equivalent to a CW-complex with one 0-cell, one 1-cell and one 2-cell. There are homotopically distinct CW-complexes with a CW-structure having those cells: the 2-cell can be attached either to the 0-cell or the 1-cell. However, in this case it is easy to see which of these CW-complexes we should choose.

Since a disc is contractible the discrete Morse function f in Example 4.5 is not optimal. However, the discrete Morse function g in the same example is optimal as it has only one critical cell.

A CW-complex always admits a discrete Morse function by assigning to each cell its dimension, but then every cell is critical so in the light of

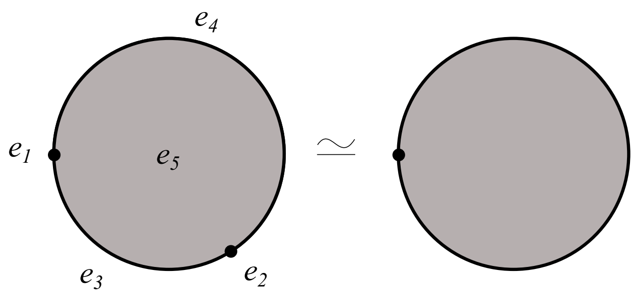

Figure 4.7

Theorem 4.10 this is a boring example. Yet, we can often find non-trivial discrete Morse functions as was the case in Example 4.5. Observe, however, that the number of cells in each dimension is not enough to determine the homotopy type of a CW-complex as was pointed out in Example 4.11. To see the correct homotopy type we would have to find out the deformations used to simplify the complex.

4.2 Relation to Simple-Homotopy Theory

As explained in the previous section, discrete Morse theory uses elementary collapses to simplify a CW-complex. This suggests that there is a strong connection between discrete Morse theory and simple-homotopy theory. Indeed, Theorem 4.10 can be stated in a stronger form, saying that a CW-complex with a discrete Morse function f is simple-homotopy equivalent to a CW-complex with as many n-cells as f has critical cells of index n. This follows from the discussion in the previous section where we saw that the only operations used to simplify a CW-complex were elementary collapses within the CW-complex. In this section we show that these internal collapses can be turned into a series of elementary expansions and collapses. This is a new result.

Proposition 4.9 allows us to give a rigorous definition for a move that is close to what we should think of as an internal collapse.

Definition 4.12. Let (X, A) be a CW-pair with $X \searrow_e A$. The homotopy-equivalence $X \cup_\varphi D^n \to A \cup_\psi D^n$, where ψ is as in Proposition 4.9, is called a *simple internal collapse* and the homotopy inverse $A \cup_\psi D^n \to X \cup_\varphi D^n$ is said to be a *simple internal expansion*.

In particular, an elementary collapse is a simple internal collapse and an elementary expansion is a simple internal expansion. It would therefore be a reasonable conjecture that allowing simple internal collapses and expansions and considering the equivalence relation generated by them might give yet another refinement of the notion of homotopy equivalence but one that is more general than simple-homotopy equivalence. However, this turns out to be false.

Lemma 4.13. *Simple internal collapse and simple internal expansion are simple-homotopy equivalences.*

Proof. The idea of the proof is in Figure 4.8. Let $\varphi : \partial D^n \to X$ be an attaching map of an n-cell before the internal collapse and let $\psi : \partial D^n \to A$ be the attaching map after. By definition we have a CW-complex $X \cup_\varphi D^n = (A \cup d^m \cup d^{m+1}) \cup_\psi D^n$ and $X \searrow_e A$. By Proposition 4.9 the maps φ and ψ are homotopic and therefore the CW-complexes $X \cup_\varphi D^n$ and $X \cup_\psi D^n$ are simple-homotopy equivalent by Theorem 3.7.

Observe that $X \cup_\psi D^n = (A \cup_\psi D^n) \cup d^m \cup d^{m+1}$ and by Proposition 4.9 $A \cup_\psi D^n$ is a well-defined CW-complex in itself. Since $d^m, d^{m+1} \notin A \cup_\psi D^n$, it follows from the hypothesis $X \searrow_e A$ that $X \cup_\psi D^n \searrow_e A \cup_\psi D^n$. In conclusion:

$$X \cup_\varphi D^n \nearrow\!\searrow X \cup_\psi D^n \searrow_e A \cup_\psi D^n.$$

$\square$

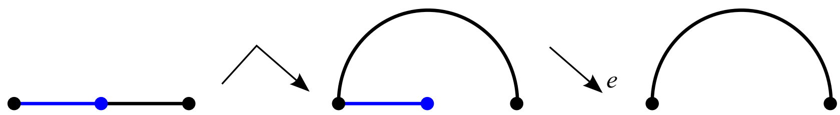

Figure 4.8 The picture illustrates the idea behind the proof of Lemma 4.13. In this case the CW-complex X consists of three 0-cells and one 1-cell (the blue one). The blue cells give the simple internal collapse (using the notation of the proof of Lemma 4.13 they are the cells d^m and d^{m+1}) and the black 1-cell is the cell that is attached first to X and then to A.

A simple internal collapse is, as the name suggests, a very simple version of how we should understand an internal collapse. If we have a CW-pair (X, A) such that $X \searrow_e A$, does it then follow that

$$X \cup \bigcup_{i \geq 1} e_i$$

and

$$A \cup \bigcup_{i \geq 1} e_i$$

are simple-homotopy equivalent? We have slightly abused notation here since we might have to change the attaching maps of some cells e_i and therefore they are not strictly speaking the same cells in both of the complexes. From now on we will not make this difference explicit. Instead, a simple internal collapse will be written $X \cup e^n \to A \cup e^n$. This is done in order to emphasize that e^n is, in essence, the same cell in both complexes. If we need to consider how the cells change we give the attaching maps explicitly, i.e. $X \cup_\varphi D^n \to A \cup_\psi D^n$.

Theorem 4.14. *Let (X, A) be a CW-pair such that $X \searrow_e A$. Then $X \cup \bigcup_{i\geq 1} e_i$ and $A \cup \bigcup_{i\geq 1} e_i$ are simple-homotopy equivalent.*

Proof. Without loss of generality we may assume that $\dim(e_i) \leq \dim(e_{i+1})$ for every $i \geq 1$. Consider the simple internal collapse $X \cup e_1 \to A \cup e_1$ and let $\varphi : \partial D_1 \to X$ be the attaching map of e_1 before the simple internal collapse and $\psi : \partial D_1 \to A$ after. We already know by Lemma 4.13 that a simple internal collapse is a simple-homotopy equivalence obtained through the following sequence of elementary expansions and collapses:

$$X \cup_\varphi D_1 \nearrow^e ((X \cup_\varphi D_1) \cup_\psi D_1) \cup e^{\dim(D_1)+1} \searrow_e X \cup_\psi D_1 \searrow_e A \cup_\psi D_1.$$

By Lemma 4.13 we can attach a cell to each of the CW-complexes in the formal deformation above without changing the simple-homotopy type of the CW-complexes involved. The result follows by induction on i. □

Definition 4.15. Let (X, A) be a CW-pair such that $X \searrow_e A$. The homotopy equivalence $X \cup \bigcup_i e_i \to A \cup \bigcup_i e_i$ is called an *elementary internal collapse* and the homotopy equivalence $A \cup \bigcup_i e_i \to X \bigcup_i e_i$ is said to be an *elementary internal expansion*. If (X, A) is a CW-pair such that $X \searrow A$, then the homotopy equivalence $X \cup \bigcup_i e_i \to A \cup \bigcup_i e_i$ is an *internal collapse* and the homotopy inverse is an *internal expansion*.

Observe that since an elementary internal collapse is a simple-homotopy equivalence, also an internal collapse is one. The same holds for internal expansion. Theorem 4.14 is what we need to restate Theorem 4.10 in a simple-homotopy form. We illustrate this with an example.

Example 4.16. Consider a two-dimensional disc with a CW-structure consisting of one 2-cell, two 1-cells and two 0-cells as in Figure 4.9. As in Example 4.5, it can be given a discrete Morse function with only one critical cell in each dimension. These two complexes are related by an internal collapse as can be seen from Figure 4.9. Therefore they are simple-homotopy equivalent by Theorem 4.14.

We now explain how to build the formal deformation from the original CW-complex to the simplified complex. The procedure we use is the one developed in the proofs in this section. First we have to consider the elementary collapse $e_1^0 \cup e_2^0 \cup e_1^1 \searrow_e e_1^0$ that gives the internal collapse. Recycling the notation in the proofs we write $A = e_1^0$ and $X = e_1^0 \cup e_2^0 \cup e_1^1 = A \cup e_2^0 \cup e_1^1$.

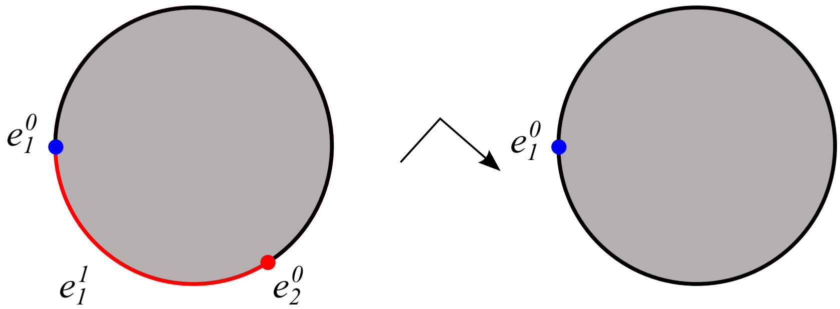

Figure 4.9 The interior collapse that we need to perform in order to get the CW-complex with a minimal amount of cells is collapsing the red cells to the blue subcomplex.

By Lemma 4.13 the CW-complexes $X \cup e^1$ and $A \cup e^1$ are simple-homotopy equivalent and from the proof of Lemma 4.13 and Theorem 3.7 we know that the formal deformation is

$$X \cup_\varphi D^1 \nearrow^e ((X \cup_\varphi D^1) \cup_\psi D^1) \cup e^2 \searrow_e X \cup_\psi D^1 \searrow_e A \cup_\psi D^1. \qquad (*)$$

To get the formal deformation from the disc with one 2-cell, two 1-cells and two 0-cells to the disc with one cell in each dimension, we must add one 2-cell to each of the CW-complexes in the formal deformation $(*)$. To see the full sequence of elementary collapses and expansions we must consider the induced simple internal collapses.

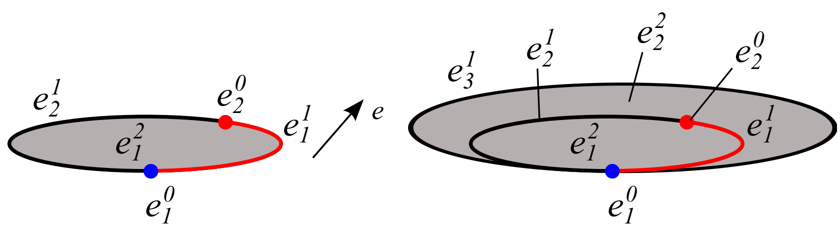

Figure 4.10 $(X \cup e_2^1) \cup e_1^2 \nearrow^e \left((X \cup e_2^1) \cup e_1^2\right) \cup e_3^1 \cup e_2^2$

We begin with the original complex. For the sake of clarity we name each cell in this example properly. The first elementary expansion inducing the first simple internal expansion is the first step of the formal deformation $(*)$. In this case the 2-cell that is attached is attached to a subcomplex not involved in the elementary expansion. The first step is the elementary expansion $(X \cup e_2^1) \cup e_1^2 \nearrow^e ((X \cup e_2^1) \cup e_3^1 \cup e_2^2) \cup e_1^2$ illustrated in Figure 4.10.

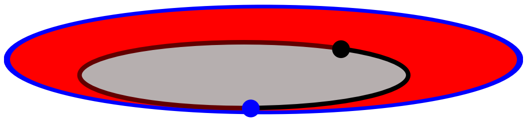

Figure 4.11

Next, we have to consider the second step of the formal deformation $(*)$. Now we do not get way with a single elementary collapse because the 2-cell we attached is attached to a 1-cell that we would like to remove in the elementary collapse as seen in Figure 4.11. So what we have to do, is to consider how a simple internal collapse is turned into a sequence of elementary collapses and expansions as in the proof of Lemma 4.13. The elementary collapse that we would like to perform within the complex is removing the cells e_2^1 and e_1^2.

We have to free the cell e_2^1 so that the cells e_2^1 and e_2^2 can be collapsed. That is, we want that the only cell attached to e_2^1 is e_2^2. After the internal collapse we must have a 2-cell attached along e_3^1, so we perform an elementary expansion where we add a 2-cell e_3^2 that is attached along e_3^1. We have to add a 3-cell as well. It is attached along e_1^2, e_3^2 and using the homotopy given by the elementary collapse $(X \cup e_2^1) \cup e_3^1 \cup e_2^2 \searrow_e X \cup e_3^1$, that is, along the cell e_2^2. This step is illustrated in Figure 4.12

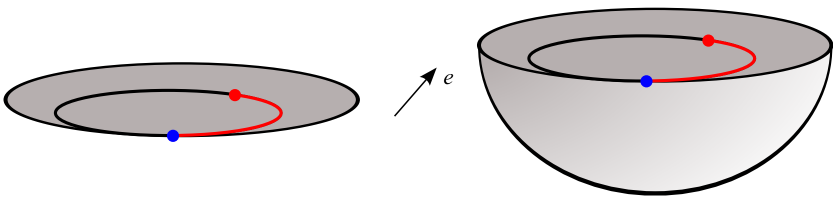

Figure 4.12 $((X \cup e_2^1) \cup e_1^2) \cup e_3^1 \cup e_2^2 \nearrow^e (((X \cup e_2^1) \cup e_1^2) \cup e_3^1 \cup e_2^2) \cup e_3^2 \cup e^3$

Now we can collapse the added 3-cell and the 2-cell e_1^2 as in Figure 4.13. The collapse frees the cell e_2^1 so we can perform the elementary collapse that we wanted to do.

We have reached the second to last CW-complex in the formal deformation $(*)$, but with one more 2-cell. As we can see from Figure 4.14 the 2-cell

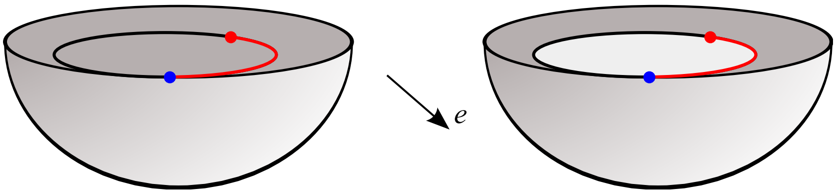

Figure 4.13 $(((X \cup e_2^1) \cup e_3^1 \cup e_2^2) \cup e_3^2) \cup e_1^2 \cup e^3 \searrow_e ((X \cup e_2^1) \cup e_3^1 \cup e_2^2) \cup e_3^2$

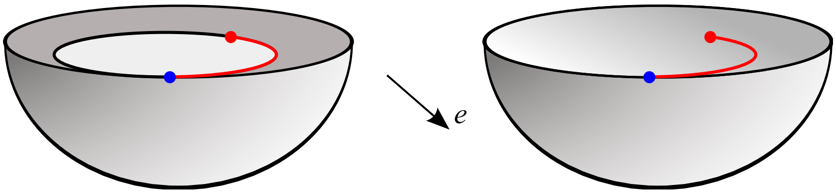

Figure 4.14 $(X \cup e_3^1 \cup e_3^2) \cup e_2^1 \cup e_2^2 \searrow_e X \cup e_3^1 \cup e_3^2$

is not attached to the cells involved in the elementary collapse we want to perform so the last step is simply the elementary collapse shown in Figure 4.15.

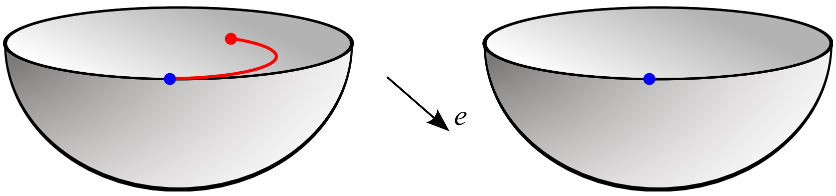

Figure 4.15 $X \cup e_3^1 \cup e_3^2 \searrow_e A \cup e_3^1 \cup e_3^2 = e_1^0 \cup e_3^1 \cup e_3^2$

Theorem 4.17. *Let X be a CW-complex with a discrete Morse function f and denote the number of critical cells of f of index n by m_n. Then X is simple-homotopy equivalent to a CW-complex with m_n cells of dimension n for each n.*

Proof. The proof is similar to the proof of Theorem 4.10 presented in [For98], but uses our new results on internal collapses in a crucial way. After having proved Lemmas 4.7 and 4.8, the theorem follows directly if the CW-complex is normal, that is if the sublevel complexes are subcomplexes. However, certain technicalities arise when considering general (finite) CW-complexes so at first we assume X to be normal.

We do not need to consider Lemma 4.7 because it is already stated in the simple-homotopy sense. In the proof of Lemma 4.8 Forman shows that the interval $(a, b] \subset \mathbb{R}$ whose preimage $f^{-1}((a, b])$ contains just one critical cell e^n of a discrete Morse function f, contains two numbers $a', b' \in \mathbb{R}$ such that $a < a' < b' < b$ and $e^n \in f^{-1}((a', b'])$. By Lemma 4.7 $X(a) \nearrow X(a')$ and $X(b') \nearrow X(b)$. Forman then shows that $X(b') = X(a') \cup e^m$, where $m = n$. But we know that $X(a) \cup e^m$ and $X(a') \cup e^m$ are simple-homotopy equivalent by Lemma 4.13 so Lemma 4.8 holds also in the simple-homotopy sense and Theorem 4.17 follows as stated here.

To generalize Theorem 4.10 to general CW-complexes Forman shows that a discrete Morse function f can always be changed to another discrete Morse function $\tilde{f}$ so that $\tilde{f}$ has exactly as many critical cells in each dimension as f. Furthermore, this new discrete Morse function avoids the technicalities - for example, every level subcomplex computed using $\tilde{f}$ is a subcomplex. Using $\tilde{f}$, the proofs of Lemmas 4.7 and 4.8 go through as they are. $\square$

Bibliography

[Cha74] Chapman, T.A. Topological invariance of whitehead torsion. *Amer. J. Math.* **96** (1974), no. 3, 488-497.

[Coh73] Cohen, M. M. *A Course in Simple-Homotopy Theory.* Graduate Texts in Mathematics, 10. Springer, 1973. 114 pp.

[DK01] Davis, J. F. and Kirk, P. *Lecture Notes in Algebraic Topology.* Graduate Studies in Mathematics, 35. American Mathematical Society, 2001. 367 pp.

[For98] Forman, R. Morse theory for cell complexes. *Adv. Math.* **134** (1998), 90-145.

[For02] Forman, R. A user's guide to discrete morse theory. *Sém. Lothar. Combin.* **48** (2002), 35 pp.

[Fra35] Franz, W. Über die torsion einer überdeckung. *J. Reine Angew. Math.* **173** (1935), 245-254.

[Fra43] Franz, W. Abbildungsklassen und fixpunktklassen dreidimensionaler linsenräume. *J. Reine Angew. Math.* **185** (1943), 65-77.

[Hat01] Hatcher, A. *Algebraic Topology.* Cambridge University Press, 2001. 556 pp.

[KL05] Kreck, M and Lück, W. *The Novikov Conjecture: Geometry and Algebra.* Oberwolfach Seminars, 33. Birkhäuser, 2005. 266 pp.

[LW69] Lundell, A. T. and Weingram, S. *The Topology of CW Complexes.* The University Series in Higher Mathematics. Van Nostrand Reinhold Company, 1969. 216 pp.

[Mag02] Magurn, B. A. *An Algebraic Introduction to K-theory.* Paperback re-issue. Encyclopedia of Mathematics and its Applications, 87. Cambridge University Press, 2002. 676 pp.

[Nic03] Nicolaescu, L. I. *The Reidemeister Torsion of 3-manifolds.* De Gruyter Studies in Mathematics, 30. Walter de Gruyter, 2003. 249 pp.

[Ran96] Ranicki, A. (ed.) *The Hauptvermutung Book.* K-Monographs in Mathematics, 1. Springer, 1996. 192 pp.

[Ran02] Ranicki, A. *Algebraic and Geometric Surgery.* Oxford Mathematical Monographs. Clarendon Press, 2002. 373 pp.

[Rue38] Rueff, M. Beiträge zur untersuchung der abbildungen von mannigfaltigkeiten. *Compositio Math.* **6** (1938), 401-487.

[Spa89] Spanier, E.H. *Algebraic Topology.* First Corrected Springer Edition. Springer, 1989. 528 pp.

[Tur01] Turaev, V. *Introduction to Combinatorial Torsions.* Springer, 2001. 123 pp.

[Whi41] Whitehead, J. H. C. On incidence matrices, nuclei and homotopy types. *Ann. of Math.* **42** (1941), 1197-1239.